

Integer Sequences of the Form $\alpha^n \pm \beta^n$

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Abstract

Suppose that we want to find all integer sequences of the form $\alpha^n + \beta^n$, where α and β are complex numbers and n is a nonnegative integer. Since $\alpha^0 + \beta^0$ is always an integer, our task is then equivalent to determining all complex pairs (α, β) such that

$$\alpha^n + \beta^n \in \mathbb{Z}, \quad n > 0. \quad (1)$$

Let p and q be two integers; and consider the quadratic equation

$$z^2 - pz - q = 0. \quad (2)$$

By the quadratic formula, the roots of (2) are

$$r = \frac{p}{2} + \frac{\sqrt{p^2 + 4q}}{2} \quad \text{and} \quad s = \frac{p}{2} - \frac{\sqrt{p^2 + 4q}}{2}. \quad (3)$$

In this paper, we prove that there is a correspondence between the roots of (2) and integer sequences of the form $\alpha^n + \beta^n$. In addition, we will show that there are no integer sequences of the form $\alpha^n - \beta^n$. Finally, we use special values of α and β to obtain a range of formulas involving Lucas and Fibonacci numbers.

1. Sums of Like Powers

In this section, we prove that the complex pair (α, β) satisfies (1) if and only if α and β are the roots of (2).

Theorem 1. *If r and s are the roots of (2), then $r^n + s^n$ is an integer for every natural number n .*

Proof. By the binomial theorem, we have

$$\begin{aligned} r^n + s^n &= \left(\frac{p}{2} + \frac{\sqrt{p^2 + 4q}}{2} \right)^n + \left(\frac{p}{2} - \frac{\sqrt{p^2 + 4q}}{2} \right)^n \\ &= \frac{1}{2^n} \sum_{i=0}^n \binom{n}{i} p^{n-i} \left(\left(\sqrt{p^2 + 4q} \right)^i + \left(-\sqrt{p^2 + 4q} \right)^i \right). \end{aligned}$$

Let $\Delta = \sqrt{p^2 + 4q}$. Then $\Delta^i + (-\Delta)^i = 0$ for odd values of i and $\Delta^i + (-\Delta)^i = 2\Delta^i$ for even values of i . Hence, replacing i by $2i$ in the last summation, the upper limit for i becomes $\lfloor n/2 \rfloor$. This yields

$$r^n + s^n = \frac{1}{2^n} \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} p^{n-2i} (2\Delta^{2i}),$$

which is equivalent to

$$r^n + s^n = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} p^{n-2i} (p^2 + 4q)^i. \quad (4)$$

Now expanding $(p^2 + 4q)^i$ in the last equation, we obtain

$$\begin{aligned} r^n + s^n &= \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} p^{n-2i} \sum_{k=0}^i \binom{i}{k} (p^2)^{i-k} (4q)^k \\ &= \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} p^{n-2i} \sum_{k=0}^i \binom{i}{k} p^{2i-2k} 2^{2k} q^k \\ &= \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} \sum_{k=0}^i \binom{i}{k} 2^{2k} p^{n-2k} q^k. \end{aligned}$$

Since $\binom{n}{2i}$ is multiplied by $\binom{i}{k}$ for $k \leq i \leq \lfloor n/2 \rfloor$, the last summation can be rearranged so that the coefficient of $p^{n-2k} q^k$ is

$$\frac{1}{2^{n-2k-1}} \sum_{i=k}^{\lfloor n/2 \rfloor} \binom{n}{2i} \binom{i}{k}, \quad (5)$$

where $0 \leq k \leq \lfloor n/2 \rfloor$. It follows that

$$r^n + s^n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1}{2^{n-2k-1}} \sum_{i=k}^{\lfloor n/2 \rfloor} \binom{n}{2i} \binom{i}{k} p^{n-2k} q^k.$$

Since p and q are integers, the proof would be completed provided one can show that (5) yields only integer values. We do so by showing that

$$2^{2k-n+1} \sum_{i=k}^{\lfloor n/2 \rfloor} \binom{n}{2i} \binom{i}{k} = \frac{n}{n-k} \binom{n-k}{k}, \quad (6)$$

where it is clear that the right-hand side of (6) is always an integer. Observe that once (6) is proved, we can write

$$r^n + s^n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k} \binom{n-k}{k} p^{n-2k} q^k. \quad (7)$$

An easy way to prove (6) is through the help of a computer algebra system. For example, the answer to the left-hand side of (6) given by *Mathematica 5.2* is

$$\frac{n\Gamma(n-k)}{\Gamma(k+1)\Gamma(n-2k+1)},$$

where Γ is the well known generalized factorial function. Using the fact that $\Gamma(n) = (n-1)!$ for the positive integer n , the answer can be written as

$$\frac{n(n-k-1)!}{k!(n-2k)!} = \frac{n}{n-k} \left(\frac{(n-k)!}{k!(n-2k)!} \right) = \frac{n}{n-k} \binom{n-k}{k},$$

and so the proof is complete¹. □

Equation (6) can be proved and thus written in many other ways. For example, Draim and Bickell Draim & Bickell (1996) proved that

$$r^n + s^n = \sum_{k=0}^{\lfloor n/2 \rfloor} \left(2 \binom{n-k}{k} - \binom{n-k-1}{k} \right) p^{n-2k} q^k. \quad (8)$$

Then using properties of binomial coefficients Koshy (2001) showed that the coefficient of $p^{n-2k} q^k$ in (8) is

$$\binom{n-k}{k} + \binom{n-k-1}{k-1} = \frac{n}{n-k} \binom{n-k}{k}.$$

Similar results are proved in (Benoumhani, 2003; Woko, 1997). More generally, Hirschhorn (2002) used summable hypergeometric series to prove, among other identities, that

$$\sum_{i=k}^{\lfloor n/2 \rfloor} \binom{n}{2i} \binom{i}{k} = 2^{n-2k-1} \left(\binom{n-k}{k} + \binom{n-k-1}{k-1} \right).$$

In fact, (6) is one of a whole class of identities involving hypergeometric series that can be proved by means of well established algorithms; see Petkovšek (1996) for a survey of such algorithms. Lastly, if $x = \sqrt{-p^2/q}$, then it can be shown that

$$r^n + s^n = 2p^n x^{-n} T_n(x/2), \quad (9)$$

where $T_n(x)$ is Chebyshev polynomial of the first kind of order n defined by the recurrence relation

$$T_0(x) = 1, \quad T_1(x) = x, \quad \text{and} \quad T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

¹Draim & Bickell (1996) proved Theorem 1 using mathematical induction. However, the direct proof given here has the advantage that its steps, as we shall see, can be used to calculate some interesting binomial sums.

The converse of Theorem 1 is also true. That is, if $\alpha^n + \beta^n$ is an integer for every natural number n , then α and β are the zeros of a quadratic polynomial with integer coefficients. Since α and β are the roots of the equation

$$z^2 - (\alpha + \beta)z + \alpha\beta = 0$$

and since $\alpha + \beta$ is an integer, the proof would be completed provided one can show that $\alpha\beta$ is also an integer. But we know that $2\alpha\beta$ is an integer since

$$2\alpha\beta = (\alpha + \beta)^2 - (\alpha^2 + \beta^2) \in \mathbb{Z}.$$

It follows that if $\alpha\beta$ is not an integer, then $\alpha\beta = m/2$ for some odd integer m . Now the fact that $\alpha^4 + \beta^4$ is an integer implies that

$$2\alpha\beta(3\alpha\beta + 2\alpha^2 + 2\beta^2) = (\alpha + \beta)^4 - (\alpha^4 + \beta^4) \in \mathbb{Z}.$$

But this could not hold unless $6\alpha^2\beta^2 = 3m^2/2$ is an integer, which is impossible when m is odd. We conclude that $\alpha\beta$ must be an integer, as required.

Finally, observe that if m is a nonzero integer, then $\alpha = m + r$ and $\beta = m + s$ are the roots of the equation

$$z^2 - (p + 2m)z - (q - pm - m^2) = 0.$$

Therefore, $\alpha^n + \beta^n$ is an integer for every positive integer n . By the binomial theorem, we have

$$\alpha^n + \beta^n = (m + r)^n + (m + s)^n = \sum_{i=0}^n \binom{n}{i} m^{n-i} (r^i + s^i) \in \mathbb{Z}. \quad (10)$$

2. Differences of Like Powers

Let $\alpha = x + iy$ and $\beta = u + iv$ be two complex numbers. Apart from the trivial case $\alpha = \beta$ and the case when both α and β are integers, we will show that there are no integer sequences of the form $\alpha^n - \beta^n$. To see this, observe that if $\alpha - \beta$ is an integer then $v = y$, and so

$$\alpha^2 - \beta^2 = (x^2 - u^2) + 2y(x - u)i.$$

It follows that $2y(x - u) = 0$, i.e., either $y = 0$ or $u = x$. But for $y = 0$, we get real α and β ; and for $u = x$, we get the trivial solution $\alpha = \beta$.

Now suppose that α and β are real numbers such that $\alpha^n - \beta^n$ is always an integer. Assuming that α and β are not both integers, then the fact that $\alpha - \beta$ is an integer implies that there exists a real number a and integers x and y such that $\alpha = x + a$ and $\beta = y + a$. It follows that

$$\alpha^2 - \beta^2 = x^2 - y^2 + 2a(x - y) \in \mathbb{Z}$$

only if a is a rational number. Clearly, a should not be an integer since otherwise α and β will be both integers. But if a is a noninteger rational number, then the same is true for $\alpha = x + a$ and $\beta = y + a$.

Theorem 2. *If α and β are two distinct (noninteger) rational numbers, then $\alpha^n - \beta^n$ cannot be an integer for every natural number n .*

Proof. The assumption that α and β are two distinct rational numbers such that $\alpha^n - \beta^n \in \mathbb{Z}$ is equivalent to saying that there exists distinct integers x , y , and z such that $|z| \neq 1$, $\gcd(x, y, z) = 1$, and

$$\frac{x^n - y^n}{z^n} \in \mathbb{Z}.$$

Since $\gcd(x, y, z) = 1$, we can divide by any common divisor of x and y until we reach $\gcd(x, y) = 1$. Now for $n = 1$, we get $z | (x - y)$. Let p be an arbitrary prime greater than the largest prime divisor of z , and define

$$P_p(x, y) = \frac{x^p - y^p}{x - y} = \sum_{i=0}^{p-1} x^{p-i-1} y^i.$$

Then $\gcd(z^p, p) = 1$ and $P_p(x, y)$ yields only integers. Since $\gcd(x, y) = 1$ and p is prime, using elementary number theory it can be easily shown that $\gcd(x - y, P_p(x, y))$ is either 1 or p . It follows that $\gcd(z, P_p(x, y)) = 1$. This coupled with the fact that $x^p - y^p = (x - y)P_p(x, y)$ implies that $z^p | (x^p - y^p)$ if and only if $z^p | (x - y)$. But this forces x to be equal to y , a contradiction of the assumption that x and y are distinct². \square

Having shown that there are no nontrivial integer sequences of the form $\alpha^n - \beta^n$, it should be mentioned that the same is not true for sequences of the form

$$\frac{\alpha^n - \beta^n}{\alpha - \beta}.$$

In particular, if we choose $\alpha = r$ and $\beta = s$, then a Fibonacci-like sequence is generated. More generally, we have the following result.

Theorem 3. *If $\alpha = y + r$ and $\beta = y + s$, then $(\alpha^m - \beta^m)/(\alpha - \beta)$ is an integer for every positive integer m .*

Proof. Since $\alpha - \beta = r - s$, the binomial theorem yields

$$\frac{\alpha^m - \beta^m}{\alpha - \beta} = \sum_{n=0}^{m-1} \binom{m-1}{n} y^{m-1-n} \frac{r^{n+1} - s^{n+1}}{r - s},$$

which is clearly an integer if $(r^{n+1} - s^{n+1})/(r - s)$ is an integer. Since $r^0 - s^0 = 0$, we need only consider positive powers of r and s . Now following an argument

²It turned out that if $x = z^m + 1$ and $y = 1$, then $(x^n - y^n)/z^n$ is an integer for $n \leq m$, where m is any positive integer. If m is prime then this is the smallest value of x that yields a solution for $n \leq m$, assuming that x and y are both positive. It is only when m is allowed to go to infinity that a solution cannot be found. In fact, it can be shown that one cannot find a positive integer N and rational numbers α and β such that $\alpha^n - \beta^n \in \mathbb{Z}$ for $n > N$, no matter how large N is.

similar to that used in Theorem 1, we get, for $n \geq 0$,

$$\begin{aligned} \frac{r^{n+1} - s^{n+1}}{r - s} &= \frac{1}{2^n} \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n+1}{2i+1} p^{n-2i} (p^2 + 4q)^i \\ &= \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{1}{2^{n-2k}} \sum_{i=k}^{\lfloor n/2 \rfloor} \binom{n+1}{2i+1} \binom{i}{k} p^{n-2k} q^k. \end{aligned}$$

Again, we can use *Mathematica* to simplify the coefficient of $p^{n-2k} q^k$ in the last equation. This yields

$$2^{2k-n} \sum_{i=k}^{\lfloor n/2 \rfloor} \binom{n+1}{2i+1} \binom{i}{k} = \frac{\Gamma(n-k+1)}{\Gamma(k+1)\Gamma(n-2k+1)}.$$

Using the properties of Γ , the right-hand side of the last equation can be written as

$$\frac{(n-k)!}{k!(n-2k)!} = \binom{n-k}{k}.$$

We conclude that

$$\frac{r^{n+1} - s^{n+1}}{r - s} = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n-k}{k} p^{n-2k} q^k. \quad (11)$$

It is clear that the right-hand side of (11) is always an integer, and so the proof is complete. \square

As before, if $(\alpha^n - \beta^n)/(\alpha - \beta)$ is an integer for every n , then α and β are the roots of a quadratic equation with integer coefficients. This is so since for $n = 2$, we get $\alpha + \beta \in \mathbb{Z}$; while for $n = 3$, we get $(\alpha + \beta)^2 - \alpha\beta \in \mathbb{Z}$, which implies that $\alpha\beta$ is an integer. Also, we can express equation (11) in terms of Chebyshev polynomials. In this case, we get

$$\frac{r^{n+1} - s^{n+1}}{r - s} = p^n x^{-n} U_n(x/2),$$

where $U_n(x)$ is Chebyshev polynomial of the second kind of order n defined by

$$U_0(x) = 1, \quad U_1(x) = 2x, \quad \text{and} \quad U_{n+1}(x) = 2xU_n(x) - U_{n-1}(x).$$

3. Special Cases

We have proved that $r^n + s^n$ is an integer for every natural number n if and only if r and s are the roots of $z^2 - pz - q = 0$. Depending on the values of p and q , some of the resulting sequences are more interesting than others. For instance, if $p = 0$ then $r = \sqrt{q}$ and $s = -\sqrt{q}$. In this case, $r^n + s^n$ is either zero (when n

is odd) or $2q^{n/2}$ (when n is even). On the other hand, if $q = 0$ then $r = p$ and $s = 0$, and thus $r^n + s^n = p^n$. So, suppose that both p and q are different from zero. Then using the identity $T_n(\theta) = \cos(n \arccos \theta)$ one can rewrite (9) as

$$r^n + s^n = 2p^n x^{-n} \cos \left(n \arccos \frac{x}{2} \right). \quad (12)$$

Since $x = \sqrt{-p^2/q}$, we see that the *simplest* sequences are obtained when both p and q are equal to one in absolute value.

First, we take $p = q = 1$. This yields $r = \phi$ and $s = -\varphi$, where $\phi = (1 + \sqrt{5})/2$ is the *golden ratio* and $\varphi = 1/\phi$. Using mathematical induction, one can easily show that $r^n + s^n = L_n$. This is the well known *Binet formula* for n -th Lucas number L_n . In fact, the formula is a special case of a more general formula that, given G_0 and G_1 , calculates the n -th *generalized* Fibonacci number defined for $n \geq 2$ by

$$G_n = pG_{n-1} + qG_{n-2},$$

where p and q are arbitrary numbers (integers in our case). It turned out that if r and s are distinct roots of $z^2 - pz - q = 0$, then

$$G_n = \frac{(G_1 - sG_0)r^n - (G_1 - rG_0)s^n}{r - s},$$

see Niven & Zuckerman (1980). In particular, if $p = q = 1$ then it easily seen that for $G_0 = 2$ and $G_1 = 1$ we get

$$G_n = \phi^n + (-\varphi)^n = L_n,$$

while for $G_0 = 0$ and $G_1 = 1$ we get

$$G_n = \frac{\phi^n - (-\varphi)^n}{\sqrt{5}} = F_n,$$

where F_n is the n -th Fibonacci number. More generally, if $G_0 = 0$ and $G_1 = 1$, then

$$G_n = \frac{r^n - s^n}{r - s}$$

for any p and q . On the other hand, if $G_0 = 2$ and $G_1 = 1$, then

$$G_n = r^n + s^n$$

for any q , provided that $p = 1$.

Beside the identity $\phi^n + (-\varphi)^n = L_n$, we can use the steps of Theorem 1 to develop other formulas for L_n . For instance, setting $p = q = 1$ in (4) yields

$$L_n = \frac{1}{2^{n-1}} \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} 5^i.$$

Doing the same in (7) we get

$$L_n = \sum_{k=0}^{\lfloor n/2 \rfloor} \frac{n}{n-k} \binom{n-k}{k}.$$

Next, we take $p = q = -1$. Then the zeros of the corresponding polynomial are

$$r = -\frac{1}{2} + \frac{i\sqrt{3}}{2} = \Lambda \quad \text{and} \quad s = -\frac{1}{2} - \frac{i\sqrt{3}}{2} = \lambda.$$

Substituting $p = q = -1$ in (12), we get $r^n + s^n = 2(-1)^n \cos(n\pi/3)$. Starting with $n = 1$, it is obvious that $r^n + s^n$ takes on the cycle $\{-1, -1, 2\}$. More generally, if we let $q = -p^2$, then we obtain

$$x = 1, \quad r = \Lambda p \quad \text{and} \quad s = \lambda p,$$

and so $r^n + s^n = 2p^n \cos(n\pi/3)$. On the other hand, $q = p^2$ gives

$$x = i, \quad r = \phi p \quad \text{and} \quad s = -\varphi p,$$

and so $r^n + s^n = L_n p^n$.

So far, we have taken $\alpha = r$ and $\beta = s$, which is equivalent to setting $m = 0$ in (10). But a whole new set of identities can be obtained by allowing m to be different from zero. Suppose that we fix $p = q = 1$. Then for $m = 1$ we get $\alpha = 1 + \phi = \phi^2$ and $\beta = 1 - \varphi = \varphi^2$. Hence, we have

$$\alpha^n + \beta^n = (\phi^2)^n + (\varphi^2)^n = \phi^{2n} + \varphi^{2n} = L_{2n}.$$

Now setting $r = \phi$, $s = -\varphi$ and $m = 1$ in the right-hand side of (10) we get

$$\sum_{i=0}^n \binom{n}{i} (\phi^i + (-\varphi)^i) = \sum_{i=0}^n \binom{n}{i} L_i = L_{2n}.$$

Moreover, since $1 + \phi$ and $1 - \varphi$ are the zeros of $z^2 - 3z + 1$, substituting $p = 3$ and $q = -1$ in (4) we obtain

$$L_{2n} = \frac{3^n}{2^{n-1}} \sum_{i=0}^{\lfloor n/2 \rfloor} \left(\frac{5}{9}\right)^i \binom{n}{2i}.$$

Doing the same in (7) yields

$$L_{2n} = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k 3^{n-2k} \frac{n}{n-k} \binom{n-k}{k}.$$

Similarly, for $m = -1$, we get $\alpha = -1 + \phi = \varphi$ and $\beta = -1 - \varphi = -\phi$. It follows that

$$\alpha^n + \beta^n = (-1)^n (\phi^n + (-\varphi)^n) = (-1)^n L_n.$$

But letting $m = -1$ in (10) gives

$$\varphi^n + (-\phi)^n = (-1)^n L_n = \sum_{i=0}^n \binom{n}{i} (-1)^{n-i} L_i.$$

Multiplying both sides of the last equation by $(-1)^n$, we deduce that

$$\sum_{i=0}^n (-1)^i \binom{n}{i} L_i = L_n.$$

Since $\binom{n}{n} = 1$, we obtain

$$\sum_{i=0}^{n-1} (-1)^i \binom{n}{i} L_i = \begin{cases} 0, & \text{if } n \text{ is even;} \\ 2L_n, & \text{if } n \text{ is odd.} \end{cases}$$

Next, we look at $m = \pm 2$. For $m = 2$, we get $\alpha = 1 + \phi^2 = \sqrt{5}\phi$ and $\beta = 1 + \varphi^2 = \sqrt{5}\varphi$. It follows that

$$\alpha^n + \beta^n = (\sqrt{5}\phi)^n + (\sqrt{5}\varphi)^n = \begin{cases} 5^k L_n, & \text{if } n = 2k; \\ 5^{k+1} F_n, & \text{if } n = 2k + 1. \end{cases}$$

This is so since

$$L_{2k} = \phi^{2k} + \varphi^{2k} \quad \text{and} \quad F_{2k+1} = \frac{\phi^{2k+1} + \varphi^{2k+1}}{\sqrt{5}}.$$

Alternatively, setting $\alpha = 1 + \phi^2$ and $\beta = 1 + \varphi^2$ in (10) we obtain

$$\alpha^n + \beta^n = \sum_{i=0}^n \binom{n}{i} (\phi^{2i} + \varphi^{2i}) = \sum_{i=0}^n \binom{n}{i} L_{2i}.$$

This leads to the known identity

$$\sum_{i=0}^n \binom{n}{i} L_{2i} = \begin{cases} 5^k L_n & \text{if } n = 2k \\ 5^{k+1} F_n & \text{if } n = 2k + 1, \end{cases}$$

which is proved in Vajda (1989). Since $\alpha = 2 + \phi$ and $\beta = 2 - \varphi$ are the zeros of $z^2 - 5z + 5$, using (4) and (7) we respectively get

$$\sum_{i=0}^n \binom{n}{i} L_{2i} = \frac{5^n}{2^{n-1}} \sum_{i=0}^{\lfloor n/2 \rfloor} \frac{1}{5^i} \binom{n}{2i} = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k 5^{n-k} \frac{n}{n-k} \binom{n-k}{k}.$$

As for $m = -2$, we obtain $\alpha = -2 + \phi = -\varphi^2$ and $\beta = -2 - \varphi = -\phi^2$. Therefore,

$$\alpha^n + \beta^n = (-\varphi^2)^n + (-\phi^2)^n = (-1)^n (\varphi^{2n} + \phi^{2n}) = (-1)^n L_{2n}.$$

Using (10) we get

$$\alpha^n + \beta^n = \sum_{i=0}^n (-2)^{n-i} \binom{n}{i} L_i = (-1)^n L_{2n}.$$

Continuing in this way, one can obtain a myriad of formulas involving Lucas numbers. Moreover, using Theorem 3, similar results involving Fibonacci numbers can be obtained as well. In addition, by expressing the roots of (2) in different forms, new sets of identities will emerge. For example, if α and β are the roots of $z^2 - z - 1 = 0$, then they can be written as $\alpha = m + r$ and $\beta = m + s$, where $m = 2$ and r and s are the roots of $z^2 + 3z + 1 = 0$. Since the roots of first equation are ϕ and $-\varphi$ and those of the second equation are $-\varphi^2$ and $-\phi^2$, we get $(2 - \varphi^2)^n + (2 - \phi^2)^n = \phi^n + (-\varphi)^n = L_n$. But substituting for α and β in (10) we obtain

$$(2 - \varphi^2)^n + (2 - \phi^2)^n = \sum_{i=0}^n (-1)^i 2^{n-i} \binom{n}{i} L_{2i} = L_n.$$

Other types of interesting identities can also be deduced. For instance, setting $k = 0$ in (6) yields

$$2^{1-n} \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} = 1 \quad \text{or} \quad \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} = 2^{n-1}. \quad (13)$$

Similarly, when $k = 1$ and $n > 1$, we get

$$2^{3-n} \sum_{i=1}^{\lfloor n/2 \rfloor} i \binom{n}{2i} = n \quad \text{or} \quad \frac{4}{n} \sum_{i=0}^{\lfloor n/2 \rfloor} i \binom{n}{2i} = 2^{n-1}. \quad (14)$$

Equating (13) with (14) we obtain

$$n \sum_{i=0}^{\lfloor n/2 \rfloor} \binom{n}{2i} = 4 \sum_{i=0}^{\lfloor n/2 \rfloor} i \binom{n}{2i}.$$

References

- Benoumhani, M. (2003). 'A sequence of binomial coefficients related to Lucas and Fibonacci numbers.' *Journal of Integer Sequences* 6(2). Available at <http://www.cs.uwaterloo.ca/journals/JIS/VOL6/Benoumhani> [5 June 2003].
- Drain, N.A. & M. Bickell (1966). 'Sums of n -th powers of roots of a given quadratic equation.' *Fibonacci Quarterly* 4: 170–178.
- Hirschhorn, M.D. (2002). 'Binomial coefficient identities and hypergeometric series.' *Australian Mathematical Society Gazette* 29: 203–208.

- Koshy, T. (2001). *Fibonacci and Lucas Numbers with Applications*. New York: John Wiley & Sons.
- Niven, I. & H. S. Zuckerman (1980). *An Introduction to the Theory of Numbers*. 4th ed. New York: John Wiley & Sons.
- Petkovšek, M., H.S. Wilf & D. Zeilberger (1996). *A = B*. Wellesley: A K Peters.
- Vajda, S. (1989). *Fibonacci and Lucas Numbers, and the Golden Section*. Chichester: Ellis Horwood.
- Woko, E.J. (1997). 'A Pascal-like triangle for $\alpha^n + \beta^n$.' *Mathematical Gazette* 81: 75–79.